Invariant theory, generalized Casimir operators, and tensor product decompositions of $U(N)$

2001 J. Phys. A: Math. Gen. 34 8237

(http://iopscience.iop.org/0305-4470/34/40/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8237–8257 PII: S0305-4470(01)11204-7

Invariant theory, generalized Casimir operators, and tensor product decompositions of $U(N)$

R T Aulwes[1], **W H Klink**[2] **and T Ton-That**[1]

[1] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
[2] Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52242, USA

Received 20 January 2000, in final form 4 December 2000
Published 28 September 2001
Online at stacks.iop.org/JPhysA/34/8237

Abstract

One of the central problems in the representation theory of compact groups concerns multiplicity, wherein an irreducible representation occurs more than once in the decomposition of the n-fold tensor product of irreducible representations. The problem is that there are no operators arising from the group itself whose eigenvalues can be used to label the equivalent representations occurring in the decomposition.

In this paper we use invariant theory along with so-called generalized Casimir operators to show how to resolve the multiplicity problem for the $U(N)$ groups. The starting point is to augment the n-fold tensor product space with the contragredient representation of interest and construct a subspace of $U(N)$ invariants. The setting for this construction is a polynomial space embedded in a Fock space of complex variables which carries all the irreducible representations of $U(N)$ (or $GL_N(\mathbb{C})$). The dimension of the invariant subspace is equal to the multiplicity occurring in the tensor product decomposition.

Generalized Casimir operators are operators from the universal enveloping algebra of outer product $U(N)$ groups that commute with the diagonal $U(N)$ action and whose eigenvalues can be used to label the multiplicity. Using the notion of dual representations we show how to rewrite the generalized Casimir operators and prove that they act invariantly on the invariant subspace. A complete set of commuting generalized Casimir operators can therefore be used to construct eigenvectors that form an orthonormal basis in the invariant subspace. Different sets of generalized commuting Casimir operators generate different orthonormal bases in the invariant subspace; the overlaps between the eigenvectors of different commuting sets of generalized Casimir operators are called invariant coefficients. We show that Racah coefficients are special cases of invariant coefficients in which the generalized Casimir operators have been chosen with respect to a definite coupling scheme in the tensor product.

The paper concludes with an example of the threefold tensor product of the eight-dimensional irreducible representation of $U(3)$ in which the multiplicity of the chosen irreducible representation is 6. Eigenvectors in the six-dimensional invariant subspace are computed for different sets of generalized Casimir operators and invariant coefficients, including Racah coefficients.

PACS numbers: 02.20.−a, 02.30.−f

1. Introduction

The study of invariants under group actions on some space has long been of interest from both a mathematical and a physical point of view. Our motivation for studying invariants is related to computing Clebsch–Gordan coefficients, wherein the tensor product of irreducible representations of a compact group is decomposed into a direct sum of irreducible representations. If bases are chosen for the irreducible representation (abbreviated irrep) spaces, Clebsch–Gordan coefficients are the overlap between the tensor product basis and direct sum basis. One of the main difficulties in computing such coefficients revolves about the multiplicity problem, when an irreducible representation occurs more than once in the direct sum decomposition. The problem is that there are no operators from the group itself whose eigenvalues can be used to break the multiplicity.

In this paper we will deal with the multiplicity problem for $U(N)$ by studying the subspace of invariants of the space of r-fold tensor products of $U(N)$ irreps augmented by the contragredient (or dual) representation of the representation of interest. Such a point of view makes use of the fact that the multiplicity of a representation λ of $U(N)$ in the tensor product $\lambda_1 \otimes \cdots \otimes \lambda_r$ is equal to the number of times the identity (or invariant) representation is contained in the augmented tensor product $\lambda_1 \otimes \cdots \otimes \lambda_r \otimes \lambda^{\surd}$, where $\lambda^{\surd}$ is the representation contragredient to λ. The dimension of the invariant subspace is equal to the multiplicity. This suggests a way not only of computing the multiplicity, but more importantly, of constructing an orthogonal basis in the invariant subspace that labels the multiplicity. In previous papers we have defined generalized Casimir operators that come from the universal enveloping algebra of the outer product group $U(N) \times \cdots \times U(N)$ and commute with the action of the diagonal subgroup. There are of course a large number of different commuting generalized Casimir operators whose eigenvalues can serve to label the multiplicity; the choice one makes depends on the way in which multiplicity is to be dealt with. But we will show how to characterize the generalized Casimir operators and give some examples that show the structure of such operators.

From this point of view Racah (or recoupling) coefficients are of particular interest, for they are independent of the bases of the irrep spaces and depend only on the basis chosen for the invariant subspace. Racah coefficients are usually defined by the ordering in which the irreps in the tensor product are coupled together; if λ_1 is tensored with λ_2, which is then tensored to λ_3, etc, a set of multiplicity labels results that differs from a different stepwise coupled set. What is in effect being done is use the eigenvalues of the Casimir operators for irreps occurring in $\lambda_1 \otimes \lambda_2$, $(\lambda_1 \otimes \lambda_2) \otimes \lambda_3$, etc to label the multiplicity. These Casimir operators are examples of generalized Casimir operators and from our point of view only one of many choices that can be made. Thus, we are generalizing the notion of Racah coefficients to include basis labels in the invariant subspace arising from the eigenvalues of any set of commuting generalized Casimir operators. We will call the overlap coefficients between different multiplicity labels invariant coefficients and reserve the term Racah coefficient to mean multiplicity labels arising from different stepwise coupling schemes. Let η_A be one basis set and η_B another. Then invariant coefficients are defined as

$$\langle \lambda\eta_A \mid \lambda\eta_B \rangle$$

where $|\lambda\eta_A\rangle$ is a (normalized) basis element in the invariant subspace of the tensor product space $V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_r} \otimes V^{\lambda^{\surd}}$.

In this paper we restrict our attention to the $U(N)$ groups, because, as will be shown, they have a particularly simple invariant structure, which makes it possible to characterize the invariant subspace in a simple fashion. The setting for analysing the invariant subspaces is a Bargmann space of square integrable holomorphic entire functions. All basis elements will be realized as polynomial functions in such spaces; in particular, basis states of $U(N)$ irreps will be realized as polynomial functions, as will tensor products. In section 2 we introduce the relevant Bargmann spaces and collect the various theorems needed to characterize invariant subspaces. Having defined the invariant subspaces, in section 3 we proceed to analyse the structure of these spaces and show how to construct bases from eigenvectors of generalized Casimir operators. If the generalized Casimir operators are chosen to give a Gelfand–Cetlin tableau, we show how to construct the corresponding Gelfand–Cetlin basis element.

To conclude, in section 4 we present several examples of bases for the invariant spaces and calculate some illustrative invariant coefficients. In particular, we show how to obtain bases that are labelled by representations of the symmetric group on r letters, in the case when the tensor product is over the same representation.

The analysis of tensor product decompositions of compact groups, especially the $U(N)$ groups, has a long history and a number of different methods have been employed to deal with the multiplicity problem. In [1] we have shown how to deal with the multiplicity problem for the $SU(N)$ groups, but the invariant theory is considerably more complicated, involving minors of determinants, and thus is not as computationally effective as the methods discussed here which generalize the ones considered in [2].

2. $U(N)$ representation theory

Let $\mathbb{C}^{n\times N}$ denote the vector space of all $n \times N$ complex matrices. If $Z = (Z_{ij})$ is an element of $\mathbb{C}^{n\times N}$, let Z^* denote its complex conjugate and write $Z_{ij} = X_{ij} + \sqrt{-1}\, Y_{ij}$; $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant N$. If $\mathrm{d}X_{ij}$ (resp. $\mathrm{d}Y_{ij}$) denotes Lebesgue measure on $\mathbb{R}$, we let $\mathrm{d}Z$ denote the Lebesgue product measure on $\mathbb{R}^{2nN}$. Define a Gaussian measure $\mathrm{d}\mu$ on $\mathbb{C}^{n\times N}$ by

$$\mathrm{d}\mu(Z) = \pi^{-nN} \exp[-\mathrm{Tr}(ZZ^{\dagger})]\, \mathrm{d}Z \tag{2.1}$$

where Tr denotes the trace of a matrix and $Z^{\dagger}$ is the transpose of Z^*.

A function $f : \mathbb{C}^{n\times N} \to \mathbb{C}$ is *holomorphic square integrable* if it is holomorphic on the entire domain $\mathbb{C}^{n\times N}$, and if

$$\int_{\mathbb{C}^{n\times N}} |f(Z)|^2\, \mathrm{d}\mu(Z) < \infty. \tag{2.2}$$

Clearly, the holomorphic square-integrable functions form a Hilbert space, the *Bargmann–Segal–Fock space*, with respect to the inner product

$$\langle f_1 \mid f_2\rangle = \int_{\mathbb{C}^{n\times N}} \overline{f_1(Z)}\, f_2(Z)\, \mathrm{d}\mu(Z). \tag{2.3}$$

Let $\mathcal{F} \equiv \mathcal{F}(\mathbb{C}^{n\times N})$ denote this Hilbert space. From [2] this inner product also can be defined by the following formula:

$$\langle f_1 \mid f_2\rangle = f_1^*(D) f_2(Z)|_{Z=0}. \tag{2.4}$$

Thus, if $f \in \mathcal{F}(\mathbb{C}^{n\times N})$ then $f(Z) = \sum_{|(\alpha)|=0}^{\infty} C_{(\alpha)} Z^{(\alpha)}$, where $(\alpha) = (\alpha_{11}, \ldots, \alpha_{nN})$, is an $n \times N$-tuple of integers $\geqslant 0$, $|(\alpha)| = \alpha_{11} + \cdots + \alpha_{nN}$, $C_{(\alpha)} \in \mathbb{C}$ and $Z^{(\alpha)} = Z_{11}^{\alpha_{11}} \cdots Z_{nN}^{\alpha_{nN}}$.

Moreover, C_α must satisfy $\sum_{|(\alpha)|=0}^{\infty}(\alpha)!|C_{(\alpha)}|^2 < \infty$, where $(\alpha)! = \alpha_{11}!\ldots\alpha_{nN}!$. For $f \in \mathcal{F}(\mathbb{C}^{n\times N})$ define f^* by

$$f^*(Z) = \sum_{|(\alpha)|=0}^{\infty} \bar{C}_{(\alpha)} Z^{(\alpha)}. \tag{2.5}$$

Then $f^*(D)$ is the differential operator obtained by formally replacing $Z_{\gamma j}$ by the partial derivative $\partial/\partial z_{\gamma j}$ $(1 \leqslant \gamma \leqslant n, 1 \leqslant j \leqslant N)$. If $f \in \mathcal{F}(\mathbb{C}^{n\times N})$ then obviously $(f^*)^* = f$ and $f^* \in \mathcal{F}(\mathbb{C}^{n\times N})$. Moreover, for all $f_1, f_2 \in \mathcal{F}(\mathbb{C}^{n\times N})$

$$\left\langle f_1^*, f_2^*\right\rangle = f_1(D) f_2^*(Z)|_{Z=0} = \sum_{|(\alpha)|=0}^{\infty} (\alpha)! C^1_{(\alpha)} \bar{C}^2_{(\alpha)} = \overline{\langle f_1, f_2\rangle} = \langle f_2, f_1\rangle. \tag{2.6}$$

Therefore, $\|f^*\| = \|f\|$ for all $f \in \mathcal{F}(\mathbb{C}^{n\times N})$. If $\mathcal{P}(\mathbb{C}^{n\times N})$ denotes the subspace of $\mathcal{F}(\mathbb{C}^{n\times N})$ of all polynomial functions in Z, then $\mathcal{P}(\mathbb{C}^{n\times N})$ is dense in $\mathcal{F}(\mathbb{C}^{n\times N})$. It follows from Weyl's 'unitarian trick' that the representation R of $U(N)$ on $\mathcal{F}$ defined by

$$(R(g)f)(Z) = f(Zg) \qquad g \in U(N) \tag{2.7}$$

is unitary.

Irreducible representations of $U(N)$ are realized on subspaces of $\mathcal{F}$ defined by

$$V^{(M)} := \left\{f \in \mathcal{F}(\mathbb{C}^{n\times N}), f(bZ) = \pi^{(M)}(b) f(Z)\right\} \tag{2.8}$$

where $b \in B_N$, the subgroup of $GL_N(\mathbb{C})$ of lower triangular matrices, and $\pi^{(M)}(b) \in \mathbb{C}$ is a representation of B defined by

$$\pi^{(M)}(b) := d_1^{(M_1)} \cdots d_n^{M_n} \tag{2.9}$$

where $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n\end{pmatrix}$ is an element of the diagonal subgroup of B and $M_1, \ldots, M_n$ satisfy the dominant condition, $M_1 \geqslant \cdots \geqslant M_n,\ n \leqslant N$.

The r-fold tensor products of irreps of $U(N)$ are also subspaces of an appropriate $\mathcal{F}$; define

$$\mathcal{H}^{(m)} = V^{(m_1)} \otimes \cdots \otimes V^{(m_r)} \tag{2.10}$$

the subspace of $\mathcal{F}(\mathbb{C}^{p\times N})$, as

$$\mathcal{H}^{(m)} = \left\{f \in \mathcal{F}(\mathbb{C}^{p\times N}) \mid f(\beta Z) = \pi^{(m)}(\beta) f(Z) = \pi^{(m_1)}(\beta_1)\cdots\pi^{(m_r)}(b_r) f(Z)\right\} \tag{2.11}$$

where $p = \sum_{i=1}^r p_i$ and β is an element of the product Borel group,

$$\beta = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_r \end{pmatrix} \tag{2.12}$$

with $b_i \in B_{p_i}$, the $p_i \times p_i$ lower triangular matrix. It follows that the outer product group $U(N) \times \cdots \times U(N)$, consisting of elements $(g_1, \ldots, g_r)$, $g_i \in U(N)$, is irreducible on $\mathcal{H}^{(m)}$, with irrep

$$\left(R_{(g_1,\ldots,g_r)} f\right)\begin{pmatrix} Z_1 \\ \vdots \\ Z_r \end{pmatrix} = f\begin{pmatrix} Z_1 g_1 \\ \vdots \\ Z_r g_r \end{pmatrix} \qquad f \in \mathcal{H}^{(m)}. \tag{2.13}$$

$(m) := (m_{11}\ldots m_{1p_1}, m_{21}\ldots m_{2p_2}, \ldots, m_{rp_r})$, that is, all the zeros in (m_i) have been deleted.

If the elements of $U(N)\times\cdots\times U(N)$ are restricted to the diagonal subgroup of all elements $(g, g, \ldots, g)$, $g \in U(N)$, which is identified with $U(N)$, the representation $R_{(g,g,\ldots,g)}$ of $U(N)$ on $\mathcal{H}^{(m)}$ becomes reducible and decomposes into a direct sum of irreducible representations of $U(N)$, with multiplicity $\mu(M)$:

$$\mathcal{H}^{(m)} = \sum_{\mu} \oplus \mu(M) V^{(M)}. \tag{2.14}$$

Rather than decomposing $\mathcal{H}^{(m)}$ directly the strategy in this paper is to adjoin the contragredient representation of (M), denoted by $(M)^{\surd}$ to $\mathcal{H}^{(m)}$ and find the invariant subspace of $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$, that is, the space of identity representations of $U(N)$. This is possible since the multiplicity $\mu(M)$ is equal to the dimension of the $U(N)$-invariant subspace of $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$ (see [1]). The appendix shows that the contragredient representation—defined with respect to linear functionals of the representation space $V^{(M)}$—can be written in the following way; consider the irrep space defined in equation (2.3) and set

$$(R^{\surd}(g) f)(Z) = f(Zg^{\surd}) \qquad f \in V^{(\mu)} \quad g \in GL_N(\mathbb{C}) \quad g^{\surd} := (g^{-1})^T. \tag{2.15}$$

Then $R^{\surd}(g)$ is equivalent to the contragredient representation.

Now let $\underbrace{GL(N,\mathbb{C}) \times \cdots \times GL(N,\mathbb{C})}_{r} \times GL(N,\mathbb{C})$ act on $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$ via the outer tensor product. If the signature (M) is $\underbrace{(M_1, \ldots, M_q, 0, \ldots, 0)}_{N}$, set

$$n = p + q \qquad Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} \in \mathbb{C}^{p\times N}$$

and $W \in \mathbb{C}^{q\times N}$, then the inner (or Kronecker) tensor product representation of $GL_N(\mathbb{C})$ on $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$ can be defined as

$$\left[R^{(m)} \otimes V^{(M)^{\surd}}(g) f\right]\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = f\left(\begin{bmatrix} Zg \\ Wg^{\surd} \end{bmatrix}\right) \tag{2.16}$$

for all $f \in \mathcal{H}^{(m)} \otimes V^{(M)^{\surd}} \subset \mathcal{F}(\mathbb{C}^{n\times N})$ and $g \in GL_N(\mathbb{C})$. Then the restriction of $R^{(m)\otimes(M)^{\surd}}$ to $U(N)$ is unitary.

In general, $GL_N(\mathbb{C})$ acts on $\mathcal{P}(\mathbb{C}^{n\times N}) \subset \mathcal{F}(\mathbb{C}^{n\times N})$ via the representation

$$[R(g) f]\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = f\left(\begin{bmatrix} Zg \\ Wg^{\surd} \end{bmatrix}\right) \qquad \forall f \in \mathcal{P}(\mathbb{C}^{n\times N}). \tag{2.17}$$

Then it follows from [3] that the ring of all polynomials in $\begin{bmatrix} Z \\ W \end{bmatrix}$, which are invariant under this action, is generated by the constants and the pq algebraically independent polynomials $P_{a\alpha}$ defined by

$$P_{a\alpha}\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = (ZW^T)_{a\alpha} = \sum_{i=1}^{N} Z_{ai} W_{\alpha i} \qquad 1 \leqslant a \leqslant p \quad 1 \leqslant \alpha \leqslant q. \tag{2.18}$$

Set $X_{a\alpha} = P_{a\alpha}\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right)$ and let X denote the $p \times q$ matrix with entries $X_{a\alpha}$. If $\mathcal{J}$ denotes the ring of all $GL_N(\mathbb{C})$-invariants, it follows that an element of $\mathcal{J}$ is a polynomial in the variable X, i.e., $f \in \mathcal{J}$ if and only if

$$f\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \varphi_f(X) \qquad X = ZW^T$$

for some polynomial $\varphi_f \in \mathcal{P}(\mathbb{C}^{p\times q})$. Note that by construction $q \leqslant \min(p, N)$, and by abuse of language if $(M) = \underbrace{(M_1, \ldots, M_q, 0, \ldots, 0)}_{p}$ let $(M)_p$ (or simply (M) if there is no possible confusion) denote the signature of the equivalent class of irreducible representations of $GL_p(\mathbb{C})$ with highest weight $\underbrace{(M_1, \ldots, M_q, 0, \ldots, 0)}_{p}$. Let $\boldsymbol{W}^{(M)_p}$ denote the vector space of all polynomial function φ in X which also satisfy the covariant condition

$$\varphi(Xb^T) = \pi^{(M)}(b)\varphi(X) \qquad \forall b \in B_q. \tag{2.19}$$

Define the representation $L^{(M)_p}$ of $GL_p(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{p\times q})$ by the equation

$$L^{(M)_p}(\gamma)\varphi(X) = \varphi(\gamma^T X) \qquad \gamma \in GL_p(\mathbb{C}). \tag{2.20}$$

Then the Borel–Weil theorem together with Weyl's 'unitarian trick' imply that the representation $L^{(M)_p}$ is irreducible with signature $(M)_p$ and its restriction to $U(p)$ is an irreducible unitary representation of the same signature.

Theorem 2.1. *If $\mathcal{J}^{(m)\otimes(M)^\surd}$ denotes the subspace of all $GL_N(\mathbb{C})$ invariant polynomials in $\mathcal{H}^{(m)\otimes(M)^\surd}$ then every element f in $\mathcal{J}^{(m)\otimes(M)^\surd}$ can be uniquely identified with an element φ_f in $\boldsymbol{W}^{(M)_p}$ which also satisfies the covariant condition:*

$$L^{(M)_p}(\beta^T)\varphi_f = \pi^{(m)}(\beta)\varphi_f$$

where β and $\pi^{(m)}(\beta)$ are defined by equations (2.11) *and* (2.12). *In other words the φ_f's constitute the subspace $(\boldsymbol{W}^{(M)_p}; \pi^{(m)})$ of $\boldsymbol{W}^{(M)_p}$ of all highest weight vectors of the restriction*

$$L^{(M)_p}|GL_{p_1}(\mathbb{C}) \times \cdots \times GL_{p_r}(\mathbb{C}).$$

Proof. Let $f \in \mathcal{J}^{(M)\otimes(M)^\surd}$ then there corresponds uniquely a function $\varphi_f \in \mathcal{P}(\mathbb{C}^{p\times q})$ such that

$$f\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \varphi_f(X) \qquad X = ZW^T.$$

The condition

$$f\left(\begin{bmatrix} I_p & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \pi^{(M)}(b) f\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right)$$

for $b \in B_q$ implies that

$$\begin{aligned}\varphi_f(Xb^T) &= \varphi_f(ZW^Tb^T) = \varphi_f(Z(bW)^T) = f\left(\begin{bmatrix} Z \\ bW \end{bmatrix}\right) \\ &= \pi^{(M)}(b) f\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \pi^{(M)}(b)\varphi_f(X)\end{aligned}$$

which means that $\varphi_f \in \boldsymbol{W}^{(M)_p}$. The condition

$$f\left(\begin{array}{c} p \\ q \end{array}\left[\begin{array}{c|c} \beta & 0 \\ \hline 0 & I_q \end{array}\right]\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \pi^{(M)}(\beta) f\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right)$$

implies that

$$\begin{aligned}L^{(M)_p}(\beta^T)\varphi_f(X) &= \varphi_f(\beta X) = f\left(\begin{bmatrix} \beta & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} Z \\ W \end{bmatrix}\right) \\ &= \pi^{(M)}(\beta) f\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \pi^{(M)}(\beta)\varphi_f(X).\end{aligned} \tag{2.21}$$

Thus, if we regard $\boldsymbol{W}^{(M)_p}$ as a $GL_{p_1}(\mathbb{C}) \times \cdots \times GL_{p_r}(\mathbb{C})$-module the condition (2.21) implies that φ_{f} is a highest weight vector of $L^{(M_p)}|_{GL_{p_1}(\mathbb{C})\times\cdots\times GL_{p_r}(\mathbb{C})}$. □

Corollary 2.2. *Let $G = U(N)$ and let $\left(R^{(M)}, \boldsymbol{V}^{(M)}\right)$ denote the irreducible unitary G-module with signature $(M) = \underbrace{(M_1, \ldots, M_q, 0, \ldots, 0)}_{N}$. Then the multiplicity of $R^{(M)}$ in $\mathcal{H}^{(m)}$ is equal to the dimension of the subspace $(\boldsymbol{W}^{(M)_p}; \pi^{(m)})$ defined in theorem* 2.1.

Proof. From theorem 2.1 of [1] this multiplicity is equal to the dimension of $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ and this vector space is isomorphic to $(\boldsymbol{W}^{(M)_p}; \pi^{(m)})$ by theorem 2.1, and the corollary follows immediately from this. □

Remark 2.3. The condition (2.21) can be broken into two parts: if β is unipotent then $L^{(M)_p}(\beta^T)\varphi_f = \varphi_f$, and if β is a diagonal matrix $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{pmatrix}$ then $L^{(M)_p}(d)\varphi_f = d_1^{m_{1,p_1}} \ldots d_p^{m_{r,p_r}}\varphi_f$. This means that φ_f are weight vectors of $\left(\boldsymbol{W}^{(M)_p}\right)$. Now the Gelfand–Cetlin tableaux provide a set of labels that can be used to get the dimension of the subspace of $(\boldsymbol{W}^{(M)_p})$ with a definite weight. It follows that a bound on the dimension of $(\boldsymbol{W}^{(M)_p}; \pi^{(m)})$ is given by the number of Gelfand–Cetlin tableaux associated with irreducible representations of $GL_p(\mathbb{C})$ of signature $(M)_p$ and with weight (m). A special case occurs when $\mathcal{H}^{(m)}$ is an r-fold tensor product of '*symmetric*' representations. (A representation of $GL_N(\mathbb{C})$ is called *symmetric* if its signature is of the form $\underbrace{(m, 0, \ldots, 0)}_{N}$, so-called because it is the space of symmetric tensors that occurs in the m-fold tensor product of the vector representation $\underbrace{(1, 0, \ldots, 0)}_{N}$ in the Schur–Weyl duality theorem, see [3, theorem 4AD]. In this special case $r = p$ and the elements β are reduced to the diagonal elements d. Thus we have also proven the following:

Corollary 2.4. *If $\mathcal{H}^{(m)}$ is a p-fold tensor product of symmetric representations of $GL_N(\mathbb{C})$ then $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ admits an orthogonal basis $\{f_\xi\}$ where f_ξ corresponds to a Gelfand–Cetlin basis element φ_ξ of $\mathcal{P}(\mathbb{C}^{p\times q})$, and ξ ranges over all Gelfand–Cetlin tableaux of $(M)_p$ with weight (m)*, i.e.,

$$f_\xi\left(\begin{bmatrix} Z \\ W \end{bmatrix}\right) = \varphi_\xi(ZW^T).$$

To explicitly construct a basis of $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ we construct a basis of $\left(\boldsymbol{W}^{(M)_p}; \pi^{(m)}\right)$. For this let $\{L_{\alpha\gamma}\}$ denote the basis of the infinitesimal operators of the left representation of $GL_p(\mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{p\times q})$ given by $L(h)\varphi(X) = \varphi(h^T X)$. Then

$$L_{\alpha\gamma} = \sum_{i=1}^{q} X_{\alpha i} \frac{\partial}{\partial X_{\gamma i}} \qquad 1 \leqslant \alpha, \gamma \leqslant p \tag{2.22}$$

and the $L_{\alpha\gamma}$ generate a Lie algebra isomorphic to $\mathcal{G}l_p(\mathbb{C})$. Moreover, $L_{\alpha\gamma}^{\dagger} = L_{\gamma\alpha}$, and the $L_{\alpha\gamma}$ with $\alpha < \gamma$ are *raising operators* while the $L_{\alpha\gamma}$ with $\alpha > \gamma$ are *lowering operators*.

If φ is a weight vector of $\left(\boldsymbol{W}^{(M)_p}\right)$ of weight (m) then

$$L(d)\varphi(X) = \varphi(dX) = d_{11}^{m_1} \ldots d_{pp}^{m_p}\varphi(X) \qquad \forall d \in D_p.$$

It follows that

$$(L_{\alpha\alpha}\varphi)(X) = \frac{\mathrm{d}}{\mathrm{dt}}(L(\exp te_{\alpha\alpha})\varphi)(X)|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\varphi\left(\left(\begin{array}{ccc} 1 & & \\ & \ddots & \vdots & \\ & --- & \exp t & \\ & & \vdots & \ddots \\ & & \vdots & & 1 \end{array}\right) X\right)\Bigg|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{m_\alpha t}\varphi(X))|_{t=0} = m_\alpha\varphi(X). \tag{2.23}$$

Using the fact that

$$[L_{\mu\nu}, L_{\alpha\beta}] = \delta_{\nu\alpha}L_{\mu\beta} - \delta_{\mu\beta}L_{\alpha\nu} \qquad 1 \leqslant \alpha, \beta, \mu, \nu \leqslant p \tag{2.24}$$

we have for $\alpha \neq \beta$

$$L_{\alpha\alpha}(L_{\alpha\beta}\varphi) = [L_{\alpha\alpha}, L_{\alpha\beta}]\varphi + L_{\alpha\beta}(L_{\alpha\alpha}\varphi) = L_{\alpha\beta}\varphi + m_\alpha L_{\alpha\beta}\varphi = (1 + m_\alpha)L_{\alpha\beta}\varphi \tag{2.25}$$

that is, $L_{\alpha\beta}$ raises the power of $d_{\alpha\alpha}$ by one, and

$$L_{\beta\beta}(L_{\alpha\beta}\varphi) = [L_{\beta\beta}, L_{\alpha\beta}]\varphi + L_{\alpha\beta}(L_{\beta\beta}\varphi) = -L_{\alpha\beta}\varphi + m_\beta L_{\alpha\beta}\varphi = (-1 + m_\beta)L_{\alpha\beta}\varphi \tag{2.26}$$

that is, $L_{\alpha\beta}$ lowers the power of $d_{\alpha\beta}$ by one. It follows immediately that

$$L(d)(L_{\alpha\beta}\varphi) = d_{11}^{m_1}\dots d_{\alpha\alpha}^{m_\alpha+1}\dots d_{\beta\beta}^{m_\beta-1}\varphi \tag{2.27}$$

that is, $L_{\alpha\beta}\varphi$ is also a weight vector of weight $(m_1, \dots, m_\alpha+1, \dots, m_\beta-1, \dots, m_p)$ if $\alpha < \beta$ and $(m_1, \dots, m_\beta - 1, \dots, m_\alpha + 1, \dots, m_p)$ if $\alpha > \beta$. And in our ordering of the weights this justifies the claim that $L_{\alpha\beta}$ is a lowering operator if $\alpha > \beta$ and is a raising operator if $\alpha < \beta$. Amongst these infinitesimal operators we have the particular operators $L_{\alpha_p\beta_p}$, where $p = p_1, \dots, p_r$, which correspond to the infinitesimal operators of the $GL_{p_i}(\mathbb{C})$ subgroup actions, $1 \leqslant i \leqslant r$. Thus, the condition $L^{(M)_p}(\beta^T)\varphi = \varphi, \varphi \in V^{(M)_p}$, β unipotent, is equivalent to the condition

$$L_{\alpha_p\beta_p}\varphi = 0 \qquad \forall\alpha_p < \beta_p \quad p = p_1, \dots, p_r. \tag{2.27}$$

By exploiting the weight changing properties of the $L_{\alpha\beta}$ we construct a set of operators $\{\tilde{\Phi}_\nu\}$, where ν ranges from 1 to the number of Gelfand–Cetlin tableaux associated with $(M)_p$ of weight (m). Each operator $\tilde{\Phi}_\nu$ is a product of lowering operators $L_{\alpha\beta}$, $\alpha < \beta$. By applying $\tilde{\Phi}_\nu$ to the highest weight vector $\varphi_{\max}^{(M)_p}$ in $\boldsymbol{W}^{(M)_p}$, where

$$\varphi_{\max}^{(M)}(X) = \Delta_1(X)^{M_1-M_2}\cdots\Delta_q X^{M_q} \tag{2.28}$$

and the Δ are principal minors, we map $\varphi_{\max}^{(M)}$ into

$$\mathcal{P}(\mathbb{C}^{p\times q})^{(m)} = \left\{p \in \mathbb{C}^{p\times q} : p(\mathrm{d}X) = \pi^{(m)}(d)p(X), \forall d \in D_p\right\}. \tag{2.29}$$

The systematic procedure for doing this, which can be implemented on a computer, makes use of the Gelfand–Cetlin tableaus for irreps $(M)_p$ and weight (m) of $U(p)$ (see [4, 5] for details).

We thus have constructed a linearly independent subspace of $\mathcal{P}(\mathbb{C}^{p\times q})$. In order that elements of this subspace belong to $(\boldsymbol{W}^{(m)}; \pi^{(m)})$ it must also satisfy the condition (2.27). This gives a set of basis elements of $(\boldsymbol{W}^{(m)}; \pi^{(m)})$ as well as the multiplicity $\mu(M)$. And this also gives us a basis for $\mathcal{J}^{(m)\otimes(M)^\surd}$. The problem of constructing orthogonal basis for $\mathcal{J}^{(m)\otimes(M)^\surd}$ is considered in section 3.

3. Orthogonal bases in $\mathcal{J}^{(m)\otimes(M)\surd}$

In the previous section, we have shown that the space of invariants $\mathcal{J}^{(m)\otimes(M)\surd}$ corresponds to the subspace $\left(\boldsymbol{W}^{(M)_p}; \pi^{(m)}\right)$ of the irreducible $U(p)$-module $W^{(M)_p}$. We also showed how to construct a (nonorthogonal) basis of $\left(\boldsymbol{W}^{(M)_p}; \pi^{(m)}\right)$, and hence of $\mathcal{J}^{(m)\otimes(M)\surd}$, by exploiting properties of the Gelfand–Cetlin tableaux associated with the weight (m). The goal of this section is to generate orthogonal bases for $\left(\boldsymbol{W}^{(M)_p}; \pi^{(m)}\right)$, or equivalently for $\mathcal{J}^{(m)\otimes(M)\surd}$ by introducing generalized Casimir operators whose eigenvalues can be used as labels of orthogonal basis vectors.

First, let us make the following observation. According to our theory of dual representations (see [6, 7]), the spectral decompositions of the pairs $(U(p), U(q))$ on $\mathcal{F}(\mathbb{C}^{p\times q})$ and $(U(p), U(N))$ on $\mathcal{F}(\mathbb{C}^{p\times N})$ are identical if $p \geqslant N$; for $p < N$ there is a one-to-one correspondence between the isotypic components with signature $(M_1, \ldots, M_p)$ in $\mathcal{F}(\mathbb{C}^{p\times p})$ and those with signature $\underbrace{(M_1, \ldots, M_p, 0, \ldots, 0)}_{N}$ in $\mathcal{F}(\mathbb{C}^{p\times N})$. This observation applied to the pairs $(U(p), U(q))$ acting on $\mathcal{F}(\mathbb{C}^{p\times q})$, $(U(p), U(N))$ acting on $\mathcal{F}(\mathbb{C}^{p\times N})$ (recall that $q \leqslant \min(p, N)$) implies that there is a correspondence between the dual modules $W^{(M)_p} \otimes V^{(M)_p}$, $W^{(M)_p} \otimes V^{(M)_q}$ and $W^{(M)_p} \otimes V^{(M)_N}$, which are the isotypic components with signature (M) in the corresponding Bargmann–Segal–Fock spaces. In particular, the highest weight vectors of the irreducible dual modules are identical if expressed in terms of the same dummy variable. It follows that the effect of the operators $\tilde{\Phi}_\nu$ on $\varphi_{\max}$, where $\tilde{\Phi}_\nu$ are expressed in terms of the infinitesimal operators

$$L^q_{\alpha\beta} = \sum_{i=1}^{q} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}} \qquad \text{or} \qquad L^N_{\alpha\beta} = \sum_{i=1}^{N} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}}$$

is identical (in fact, the global action $L(h), h \in U(p)y$, is always the same) on $\mathcal{F}(\mathbb{C}^{p\times q})$, $\mathcal{F}(\mathbb{C}^{p\times p})$ or $\mathcal{F}(\mathbb{C}^{p\times N})$. But the operators $\tilde{\Phi}_\nu$, if expressed in terms of the $L^N_{\alpha\beta}$, are exactly the linearly independent intertwining operators that map the $U(N)$ irreducible module $\boldsymbol{V}^{(M)}$ into the tensor product $\mathcal{H}^{(m)}(\mathbb{C}^{n\times N})$. This is exactly the problem we considered in [2].

The procedure by which generalized Casimir operators are used to break the multiplicity is quite general. Let (G', G) and (H', H) be two pairs of dual (representation) modules acting on $\mathcal{F}(\mathbb{C}^{n\times N})$ in such a way that G is a closed subgroup of H and H' is a closed subgroup of G'. Let $\mathcal{W}_{n\times N}$ denote the Weyl algebra of all differential operators with polynomial coefficients on $\mathbb{C}^{n\times N}$. Let U_G, $U_{G'}$, U_H and $U_{H'}$ denote the universal algebras (of the representations) of G, G', H and H', respectively. Then all these algebras are subalgebras of $\mathcal{W}_{n\times N}$. If $\mathcal{Z}(\mathcal{U}_G; \mathcal{W}_{n\times N})$, $\mathcal{Z}(\mathcal{U}_{G'}; \mathcal{W}_{n\times N})$, $\mathcal{Z}(\mathcal{U}_H; \mathcal{W}_{n\times N})$ and $\mathcal{Z}(\mathcal{U}_{H'}; \mathcal{W}_{n\times N})$ denote the centralizers of $\mathcal{U}_G, \mathcal{U}_{G'}, \mathcal{U}_H$ and $\mathcal{U}_{H'}$ in $\mathcal{W}_{n\times N}$ then for many dual representations $\mathcal{Z}(\mathcal{U}_G; \mathcal{W}_{n\times N}) = \mathcal{U}_{G'}$, $\mathcal{Z}(\mathcal{U}_{G'}; \mathcal{W}_{n\times N}) = \mathcal{U}_G$, $\mathcal{Z}(\mathcal{U}_H; \mathcal{W}_{n\times N}) = \mathcal{U}_{H'}$ and $\mathcal{Z}(\mathcal{U}_{H'}; \mathcal{W}_{n\times N}) = \mathcal{U}_H$.

Definition 3.1. *Let ρ_H be a unitary representation of a Lie group H on a Hilbert space $\mathcal{H}$, let G be a closed subgroup of H. Let $\mathcal{U}_H$ (resp. $\mathcal{U}_G$) denote the universal enveloping algebra generated by the infinitesimal action of ρ_H (resp. $\rho_G = \rho_{H|G}$). An element $C \in \mathcal{U}_H$ that commutes with $\mathcal{U}_G$ is called a generalized Casimir operator for the pair (ρ_H, ρ_G) (or simply (H, G)).*

Such operators are useful not only for compact groups but also for more general classes of groups, including semidirect product groups such as the Poincaré or Galilei groups, where it is known how to construct sets of generalized commuting operators whose eigenvalues label the invariant subspaces.

Theorem 3.2. *Under the assumption that* (H', H) *and* (G', G) *are two dual (representations) modules acting on* $\mathcal{F}(\mathbb{C}^{n\times N})$ *such that G is a closed subgroup of H and H′ is a closed subgroup of G′, if* $\mathcal{C}_H(G)$ *(resp.* $\mathcal{C}_{G'}(H')$*) denotes the set of generalized Casimir operators for* (H, G) *(resp.* (G', H')*) then* $\mathcal{C}_H(G) = \mathcal{C}_{G'}(H')$*.*

Proof. Let $\mathcal{C} \in \mathcal{C}_H(G)$ then since $\mathcal{C}$ commutes with $\mathcal{U}_G$ it belongs to $\mathcal{Z}(\mathcal{U}_G; \mathcal{W}_{n\times N}) = \mathcal{U}_{G'}$. On the other hand, $\mathcal{C} \in \mathcal{U}_H = \mathcal{Z}(\mathcal{U}_{H'}; \mathcal{W}_{n\times N})$ must commute with $\mathcal{U}_{H'}$. Thus $\mathcal{C} \in \mathcal{C}_{G'}(H')$, and hence $\mathcal{C}_H(G) \subset \mathcal{C}_{G'}(H')$. Similarly, we have the inclusion $\mathcal{C}_{G'}(H') \subset C_H(G)$, and thus $\mathcal{C}_H(G) = \mathcal{C}_{G'}(H')$. □

Now if λ_i denotes an equivalence class of the irreducible representation of the group G on the space V^{λ_i}, $1 \leqslant i \leqslant n$, then $V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_n}$ is an irreducible $\underbrace{G \times \cdots \times G}_{n} = H$-module. On the restriction to the diagonal subgroup which is identified with G the Kronecker tensor product G-module $V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_n}$ becomes reducible and in general multiplicity occurs. Generalized Casimir operators may then be used to break this multiplicity.

In the context of our problem let $\underbrace{U(N) \times \cdots \times U(N)}_{n}$, or equivalently, $GL_N(\mathbb{C}) \times \cdots \times GL_N(\mathbb{C}) = H$ act on $\mathcal{H}^{(m)}$. Let $G = GL_N(\mathbb{C})$ and $\mathcal{U}_H$ (resp. $\mathcal{U}_G$) denote the universal enveloping algebra of the infinitesimal action, then $\mathcal{U}_H = \mathcal{U}(\mathcal{G}\times\cdots\times\mathcal{G}) \cong \mathcal{U}(\mathcal{G})\otimes\cdots\otimes\mathcal{U}(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra generated by the infinitesimal action of G on $\mathcal{H}^{(m)}$. The set of generalized Casimir operators $\mathcal{C}_H(G)$ is generated by the differential operators of the form

$$\mathrm{Tr}\left[\left[R^{(p_1)}\right]^{d_1} \cdots \left[R^{(p_r)}\right]^{d_r}\right] \tag{3.1}$$

where the matrices $R^{(p_i)}$, $1 \leqslant i \leqslant r$, have (j, k) entry:

$$R_{jk} = \sum_{\alpha=1}^{p} Z_{\alpha j} \frac{\partial}{\partial Z_{\alpha k}} \qquad 1 \leqslant j, k \leqslant N. \tag{3.2}$$

The d_i are integers $\geqslant 0$ (see [2, property 3.3]), and 'Tr' denotes the *noncommutative* trace operator. Moreover, as shown in [2, property 3.5], these generalized Casimir operators are Hermitian.

To see how these generalized Casimir operators act on $\mathcal{J}^{(m)\otimes(M)\surd}$, and also for computational purposes, it is more convenient to use the dual representation and theorem 3.2 to compute $\mathcal{C}_H(G) = \mathcal{C}'_G(H')$ in terms of the dual actions of H and G on $\mathcal{F}(\mathbb{C}^{p\times N})$. The dual action of H on $\mathcal{F}(\mathbb{C}^{p\times N})$ is defined by

$$L\left(\begin{bmatrix} \boxed{g'_1} & & & \bigcirc \\ & \boxed{g'_2} & & \\ & & \ddots & \\ \bigcirc & & & \boxed{g'_r} \end{bmatrix}\right) f(x) = f\left(\begin{bmatrix} \boxed{g'^T_1} & & & \\ & \boxed{g'^T_2} & & \\ & & \ddots & \\ & & & \boxed{g'^T_r} \end{bmatrix} X\right) \tag{3.3}$$

for all $g'_i \in GL_{p_i}(\mathbb{C})$, $1 \leqslant i \leqslant r$, and for all $f \in \mathcal{F}(\mathbb{C}^{p\times N})$. The dual action of G on $\mathcal{F}(\mathbb{C}^{p\times N})$, $p = p_1 + \cdots + p_r$, is given by

$$[L(g')f](X) = f((g')^T X) \qquad g' \in GL_p(\mathbb{C}) \tag{3.4}$$

and thus $H' = GL_{p_1}(\mathbb{C}) \times \cdots \times GL_{p_r}(\mathbb{C})$. The Lie algebra of the infinitesimal action of G' is generated by the vector fields

$$L_{\alpha\beta} = \sum_{i=1}^{N} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}} \qquad 1 \leqslant \alpha, \beta \leqslant p$$

and the universal enveloping algebra $\mathcal{U}_{G'}$ is particularly simple. If we write the matrix $[L] = (L_{\alpha\beta})$, $1 \leqslant \alpha, \beta \leqslant p$, in block form as

$$[L] = \begin{bmatrix} [L]_{11} & \dots & [L]_{1r} \\ \vdots & & \\ [L]_{r1} & \dots & [L]_{rr} \end{bmatrix} \tag{3.5}$$

then, as was shown in [2, theorem 3.1], $\mathcal{C}_{G'}(H')$ is generated by the generalized Casimir operators of the form

$$\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\dots[L]_{u_ku_1}) \qquad 1 \leqslant u_j \leqslant r \quad 1 \leqslant j \leqslant k. \tag{3.6}$$

The Hermitian operators formed from these generalized Casimir operators were used in [2] to break the multiplicity in the tensor product decomposition of $\mathcal{H}^{(m)}$. But as remarked earlier in this section, in the construction of a nonorthogonal basis $\left(\boldsymbol{W}^{(M)_p}; \pi^{(n)}\right)$, this basis is obtained by applying the maps $\tilde{\Phi}_\nu$ to $\varphi^{(M)}_{\max}$ and then requiring that they satisfy condition (2.27). Further, as remarked earlier $\tilde{\Phi}_\nu$ can be expressed equivalently in terms of $L^q_{\alpha\beta}$ or $L^N_{\alpha\beta}$. And the condition (2.27) can be expressed as

$$L_{\alpha_p\beta_p}\varphi = 0 \qquad \forall \alpha_p < \beta_p \quad p = p_1, \dots, p_r \tag{3.7}$$

where

$$L_{\alpha_p\beta_p} = \sum_{i=1}^{N} Z_{\alpha_p i}\,\frac{\partial}{\partial Z_{\beta_p i}}$$

instead of $\sum_{i=1}^{q} Z_{\alpha_p i}\,\partial/\partial Z_{\beta_p i}$. But these are part of the infinitesimal operators of the action of H'. It follows that if Φ_μ are obtained from $\tilde{\Phi}_\nu$ by applying condition (2.27), then for $C \in \mathcal{C}_{G'}(H') = \mathcal{C}_H(G)$, C commutes with Φ_μ. Indeed, $\tilde{\Phi}_\nu$ maps $V^{(M)}$ into $\mathcal{P}^{(m)}$, and C commuting with H' implies that C commutes with Φ_μ. We summarize the above results in the following:

Proposition 3.3. *The generalized Casimir operators given by equation* (3.6) *leaves the subspace* $(\boldsymbol{W}^{(M)_p}; \pi^{(m)})$, *or equivalently,* $\mathcal{J}^{(m)\otimes(M)^\surd}$, *invariant.*

Assume now that a set of generalized commuting Hermitian operators $\{C_t\}$ has been chosen such that

$$C_t\Phi_\mu\varphi^{(M)}_{\max} = \Phi_\mu C_t\varphi^{M}_{\max} \tag{3.8}$$

that is, each C_t leaves the space $\left(\boldsymbol{W}^{(M)}; \pi^{(m)}\right)$ invariant. Since $\{C_t\}$ is a commuting set of Hermitian operators on $\left(\boldsymbol{W}^{(M)}; \pi^{(m)}\right)$ they can be simultaneously diagonalized; calling the eigenvalues η, then the set $\{\eta\}$ may be used to label an orthogonal basis of $\left(\boldsymbol{W}^{(M)}; \pi^{(m)}\right)$, and hence of $\mathcal{J}^{(m)\otimes(M)^\surd}$. Examples will be given in section 4.

4. *U*(3) examples

To illustrate the methods developed in previous sections, we give below two examples. They are kept relatively simple but nevertheless typical of the general process of obtaining the space of invariants $\left(\boldsymbol{W}^{(M)}; \pi^{(m)}\right)$ (or equivalently $\mathcal{J}^{(m)\otimes(M)^\surd}$), the Gelfand–Cetlin basis for a given weight (m) of an irreducible unitary representation of $U(N)$ with signature (M), the multiplicty $\mu(M)$ of (M) in the tensor product (m), the generalized Casimir operators, the Clebsch–Gordan and the invariant coefficients involved. For details of this process which includes many computer programs see [4, 5].

Example 1. Symmetric. For our first example we consider the special case where $\mathcal{H}^{(m)}$ is an r-fold tensor product of *symmetric* representations (see corollary 2.4).

Let $N = 3$ and let $(m_1) = (2, 0, 0)$, $(m_2) = (2, 0, 0)$ and $(m_3) = (3, 0, 0)$ denote the signatures of three symmetric irreducible unitary representations of $U(3)$. Then $(m) = (2, 2, 3)$ and $r = p = 3$. Choose $(M) = (4, 2, 1)$ then $q = 3$ and $(M)^{\surd} = (-1, -2, -4)$. The Gelfand–Cetlin tableaux associated with the signature (M) and weight (m) are

$$\begin{pmatrix} 4 & & 2 & & 1 \\ & k_1 & & k_2 & \\ & & \ell & & \end{pmatrix} \qquad \text{where} \qquad \begin{cases} \ell = 2 \\ k_1 + k_2 - \ell = 2 \\ 7 - (k_1 + k_2) = 3 \end{cases}$$

and $4 \geqslant k_1 \geqslant 2 \geqslant k_2 \geqslant 1$, $k_1 \geqslant \ell \geqslant k_2$. Obviously, the only two solutions are

$$\begin{pmatrix} 4 & & 2 & & 1 \\ & 2 & & 2 & \\ & & 2 & & \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 2 & & \end{pmatrix}. \tag{4.1}$$

Thus the multiplicity $\mu(M)$ is 2, and therefore $\mathcal{J}^{(m)} \otimes (M)^{\surd}$ $\left(\text{or equivalently } \left(\boldsymbol{W}^{(M)}; \pi^{(m)}\right)\right)$ has dimension 2. The two left intertwining operators are $\tilde{\Phi}_1 = L_{21}L_{32}L_{31}$ and $\tilde{\Phi}_2 = L_{31}^2$. The result of applying these operators to the highest weight vector

$$\varphi_{\max}^{(M)}(X) = \left(\Delta_1^1(X)\right)^2 \Delta_{12}^{12}(X)\Delta_{123}^{123}(X)$$

is

$$\tilde{f}_1 = 2\left(\Delta_1^3(X)\right)^2 \Delta_{12}^{12}(X)\Delta_{123}^{123}(X) - 4\Delta_1^1(X)\Delta_1^3(X)\Delta_{12}^{23}(X)\Delta_{123}^{123}(X)$$
$$\tilde{f}_2 = 2\Delta_1^2(X)\Delta_1^3(x)\Delta_{12}^{13}(X)\Delta_{123}^{123}(X) + 2\Delta_1^1(X)\Delta_1^3(X)\Delta_{12}^{23}(X)\Delta_{123}^{123}(X).$$

The signatures in the coupling chain corresponding to the subgroups that produce the final signature (4, 2, 1) are from equation (4.1)

$$(2, 2) \to (4, 2, 1)$$
$$(3, 1) \to (4, 2, 1).$$

To identify these signatures of $U(2)$ in the intermediate coupling it suffices to use the quadratic Casimir operator

$$C_{12} = \mathrm{Tr}\left(\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\right)^2.$$

C_{12} leaves the subspace spanned by $\tilde{f}_1$ and $\tilde{f}_2$ invariant. Since C_{12} is Hermitian its action on this two-dimensional subspace results in two orthogonal eigenvectors which correspond to the eigenvalues 8 and 12, respectively,

$$f_1 = \tilde{f}_1 + \tilde{f}_2 \qquad \text{and} \qquad f_1 = \tilde{f}_2. \tag{4.2}$$

By normalizing f_1 and f_2 we obtain the two Gelfand–Cetlin basis elements which correspond to the tableaus $\begin{pmatrix} 4 & & 2 & & 1 \\ & 2 & & 2 & \\ & & 2 & & \end{pmatrix}$ and $\begin{pmatrix} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 2 & & \end{pmatrix}$. By substituting ZW^T for X in equation (4.2) we also obtain an orthogonal basis for $\mathcal{J}^{(m)\otimes(M)^{\surd}}$.

Note that, in general, unless one is interested in finding $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ explicitly, in the process of finding Gelfand–Cetlin bases, the dummy variable X plays no role, and thus it can be dropped altogether in all computer programs.

Example 2. U(3), nonsymmetric. Consider the 3-fold tensor product $(2,1,0)\otimes(2,1,0)\otimes(2,1,0)$. Then $(m) = (2,1,2,1,2,1)$ and $p = 6$. We want to find the equivalent copies of the irreducible representation with signature $(5,3,1)$. Then $(M) = (5,3,1)$, $(M)^{\surd} = (-1,-3,-5)$ and $q = N = 3$. What is particularly interesting about this example is that multiplicity already appears in any of the intermediate couplings $(2,1,0)\otimes(2,1,0)$. This phenomenon of multiplicity occurring in intermediate couplings never appears in the symmetric case. The task is to find a set of commuting generalized Casimir operators that will break the multiplicities. Unlike the symmetric case, there are no prescriptions in general for finding a set of commuting generalized Casimir operators that will break multiplicities for all cases.

In this example, there are 36 intertwining operators that map the highest weight vector in $V^{(5,3,1)}$ into $\mathcal{P}^{(2,1,2,1,2,1)}$. Since there are six solutions to the Borel condition, the multiplicity is 6. Denote these six polynomials as $f_1,\ldots,f_6$.

If

$$C_1 = \mathrm{Tr}([L]_{12}[L]_{22}[L]_{21})$$
$$C_2 = \mathrm{Tr}([L]_{13}[L]_{33}[L]_{31}) + \mathrm{Tr}([L]_{23}[L]_{33}[L]_{32})$$

where the 6×6 matrix $[L]$ is partitioned into 2×2 blocks, then C_1 and C_2 are commuting generalized Casimir operators. The matrices representing their actions in the subspace $W_0 \equiv \left(\boldsymbol{W}^{(M)};\pi^{(m)}\right)$ spanned by $f_1,\ldots,f_6$ are, respectively,

$$A_1 = \begin{pmatrix} 29 & 7 & \frac{7}{3} & -\frac{13}{3} & -3 & 0 \\ -25 & 43 & \frac{19}{3} & -\frac{16}{3} & -3 & 0 \\ -\frac{141}{2} & \frac{123}{2} & \frac{117}{2} & -\frac{51}{2} & -\frac{63}{2} & 0 \\ -144 & 63 & 39 & 3 & -24 & 0 \\ -\frac{99}{2} & \frac{99}{2} & \frac{33}{2} & -\frac{33}{2} & \frac{27}{2} & 0 \\ 237 & 69 & -57 & 39 & 9 & 42 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 50 & -23 & \frac{5}{3} & 3 & 3 & -\frac{4}{3} \\ \frac{43}{2} & 64 & -\frac{16}{3} & 3 & 3 & \frac{2}{3} \\ \frac{9}{2} & -72 & 52 & 12 & 36 & -2 \\ 126 & 9 & -35 & 78 & 27 & 4 \\ 18 & -45 & -9 & 9 & 84 & 0 \\ -225 & -126 & 50 & -24 & -6 & 50 \end{pmatrix}$$

Diagonalizing A_1 in the subspace results in four one-dimensional and one two-dimensional eigenspaces. C_2 breaks the degeneracy and the eigenvalues of the simultaneous eigenvectors of C_1 and C_2 are

$$(C_1, C_2) = \left[\left(\tfrac{2}{3}(13\pm\sqrt{5}), 75\right), (30, 66), (36, 57), \left(42, \tfrac{1}{2}(105\pm\sqrt{105})\right)\right].$$

Since the signatures $(2,1,0)$ in the 3-fold tensor product are identical, we can use the embedded left action of the symmetric group, S_2, to break the multiplicity in the intermediate couplings. The signatures in the $((1,2),3)$ coupling that contribute to the final signature $(5,3,1)$ are $(3,2,1)$, $(3,3,0)$ and $(4,2,0)$. Using the usual quadratic and cubic Casimir operators of $GL(4,\mathbb{C})$, we find the polynomials that transform according to these signatures:

$(3,2,1,0)$

$$h_1 = -\tfrac{2}{75} f_1 - \tfrac{13}{50} f_2 - \tfrac{1}{25} f_3 - \tfrac{21}{50} f_4 + f_6$$
$$h_2 = \tfrac{52}{225} f_1 + \tfrac{19}{225} f_2 + \tfrac{43}{25} f_3 + \tfrac{16}{50} f_4 + f_5$$

(3, 3, 0, 0)

$$h_3 = \frac{8}{9} f_1 + \frac{5}{9} f_2 + 5 f_3 + 2 f_4 + 2 f_5 - 5 f_6$$

(4, 1, 1, 0)

$$h_4 = \frac{1}{9} f_1 - \frac{1}{18} f_2 + \frac{1}{2} f_3 + f_6$$

(4, 2, 0, 0)

$$h_5 = f_6$$
$$h_6 = -\frac{1}{3} f_1 + \frac{1}{6} f_2 - \frac{1}{2} f_3 + f_4.$$

Note that we have two two-dimensional degenerate subspaces. The signature (3, 2, 1) occurs with multiplicity 2 in $(2, 1, 0) \otimes (2, 1, 0)$ and the left action of S_2 which permutes the 2×3 blocks Z_1 and Z_2 in $f\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$ can be used to break the multiplicity. In particular, the +1 and -1 eigenvectors are

$$h_{|(1,2,3),+\rangle} = -\frac{4}{3} h_1 + h_2$$
$$h_{|(1,2,3),-\rangle} = -8 h_1 + h_2$$

The signature (4, 2, 0) occurs with multiplicity 1 in $(2, 1, 0) \otimes (2, 1, 0)$. The reason that we have two vectors that transform as (4, 2, 0) is that in the final coupling $(4, 2, 0) \otimes (2, 1, 0)$, (5, 3, 1) occurs twice. The generalized Casimir operator, C_2, from above commutes with the quadratic and cubic Casimir operators and breaks the degeneracy with eigenvalues $\frac{1}{2}(105 \pm \sqrt{105})$.

As an example of computing Racah coefficients, apply the above procedures to the (1, (2, 3)) coupling. The unnormalized polynomials associated with the basis vectors $|(1, (2, 3)), \pm\rangle$ that transform as (3, 2, 1) are

$$h_{|(1,(2,3)),+\rangle} = -\frac{1}{9} f_1 - \frac{7}{36} f_2 - \frac{7}{6} f_3 - \frac{5}{6} f_4 - \frac{5}{12} f_5 + f_6$$
$$h_{|(1,(2,3)),-\rangle} = -\frac{1}{6} f_1 - \frac{7}{24} f_2 - \frac{7}{4} f_3 - \frac{7}{8} f_4 - \frac{7}{8} f_5 + f_6.$$

Then the Racah coefficients are

	$h_{\|(1,(2,3)),+\rangle}$	$h_{\|(1,(2,3)),-\rangle}$
$h_{\|((1,2),3),+\rangle}$	$\frac{1}{5}$	0
$h_{\|((1,2),3),-\rangle}$	0	$\frac{1}{3}$.

It is also possible to bring in the full S_3 group action to break the multiplicity. S_3 acts by permuting the three 2×3 blocks $\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$. If σ is the matrix form of an element in S_3 embedded in $GL(6, \mathbb{C})$, then the action is defined as

$$(\sigma \cdot f)\left(\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}\right) = f\left(\sigma^{-1}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}\right).$$

Computing the characters for the permutations (1, 2) (3) and (1, 3, 2) which represent the two conjugacy classes shows that the subspace W_0 contains the symmetric and antisymmetric representations of S_3 and two copies of a two-dimensional irreducible representation of S_3. To distinguish between these two copies, we use the following self-adjoint generalized Casimir operator which commutes with S_3 (see [8] for these generalized Casimir operators):

$$T = \sum_{\sigma \varepsilon S_3} \mathrm{Tr}\left([L]_{\sigma(1)\sigma(2)}[L]_{\sigma(2)\sigma(3)}[L]_{\sigma(3)\sigma(2)}\right).$$

The symmetric and antisymmetric representations are given by

$$\begin{aligned} h_S &= -4f_1 - f_2 - 18f_3 - 6f_4 - 9f_5 + 12f_6 \\ h_A &= -20f_1 - 17f_2 - 114f_3 - 60f_4 - 51f_5 + 168f_6. \end{aligned}$$

The eigenvectors of T that label the two two-dimensional irreducible representations of S_3 are $\lambda_1 = 2(263 - \sqrt{457})$ and $\lambda_2 = 2(263 + \sqrt{457})$.

Let

$$\begin{aligned} h_{1,+} &= \tfrac{2}{3} f_1 + \tfrac{7}{3} f_3 + f_5 - \tfrac{14}{3} f_6 \\ h_{1,-} &= -\tfrac{7}{3} f_1 - \tfrac{1}{3} f_2 - \tfrac{29}{2} f_3 + 2f_4 - \tfrac{11}{2} f_5 + f_6 \\ h_{2,+} &= -\tfrac{1}{3} f_1 + \tfrac{1}{6} f_2 + f\tfrac{1}{2} f_3 + f_4 + 2f_6 \\ h_{2,-} &= -\tfrac{1}{6} f_1 - \tfrac{1}{12} f_2 - f\tfrac{5}{4} f_3 - f_4 + \tfrac{1}{2} f_5 + f_6 \\ f_{1,+} &= \tfrac{3}{544}(109 - \sqrt{456})h_{1,+} + h_{2,+} \\ f_{1,-} &= \tfrac{1}{544}(109 - \sqrt{456})h_{1,+} + h_{2,-} \\ f_{2,+} &= \tfrac{3}{544}(109 + \sqrt{456})h_{1,+} + h_{2,+} \\ f_{2,-} &= \tfrac{1}{544}(109 + \sqrt{456})h_{1,-} + h_{2,-}. \end{aligned}$$

Then

$$\begin{aligned} &T f_{i,\pm} = \lambda_i f_{i,\pm} \\ &\sigma_{(1,2)(3)} \cdot f_{i,\pm} = \pm f_{i,\pm} \\ &\sigma_{(1,2,3)} \cdot f_{i,+} = -\tfrac{1}{2} f_{i,+} + f_{i,-} \\ &\sigma_{(1,2,3)} \cdot f_{i,-} = \tfrac{3}{4} f_{i,+} - \tfrac{1}{2} f_{i,-}. \end{aligned}$$

Hence, $\{f_{1,\pm}\}$ and $\{f_{2,\pm}\}$ span the two two-dimensional irreducible representations of S_3 and each basis vector is uniquely labelled by the eigenvalues of S_2 and T.

In table 1 we list the invariant coefficients of two different schemes of multiplicity breaking; namely, the scheme using the generalized Casimir operators (C_1, C_2) versus the scheme using the chain of groups $S_2 \subset S_3$ together with the generalized Casimir operator T.

Table 1.

		S_2, S_3, T				
(C_1, C_2)	$\{h_s\}$	$\{h_A\}$	$\{f_1+\}$	$\{f_1-\}$	$\{f_2+\}$	$\{f_2-\}$
$\left\{\frac{2}{3}(13 - \sqrt{5}), 75\right\}$	−0.657 7840	0.074 6542	0.004 845 37	−0.429 3960	−0.410 875	0.613 842 40
$\left\{\frac{2}{3}(13 + \sqrt{5}), 75\right\}$	0.311 8570	0.245 7010	−0.345 481 00	0.024 4645	0.830 279	0.167 742 00
{30, 66}	0.036 0342	0.353 8480	−0.430 615 00	−0.239 2890	0.793 845	0.146 345 00
{36, 57}	−0.252 9720	−0.310 1540	0.390 077 00	0.154 6000	−0.805 737	0.218 478 00
$\left\{42, \frac{1}{2}(105 - \sqrt{105})\right\}$	−0.300 5870	−0.276 6197	0.366 665 00	0.067 1861	−0.801 773	0.101 521 00
$\left\{42, \frac{1}{2}(105 + \sqrt{105})\right\}$	−0.189 0600	−0.316 9780	0.405 498 00	0.151 2410	−0.818 345	−0.008 916 07

5. Conclusion

The results that have been obtained in this paper for resolving the $U(N)$ multiplicity problem with eigenvalues of generalized Casimir operators are actually part of a much more general set-up. Let (π, V) be an irreducible module for a group H and consider the restriction to a subgroup G of H. Then $\pi \mid G$ is in general reducible and may involve multiplicity. From the universal enveloping algebra of H, $\mathcal{U}(H)$, form the generalized Casimir operators, those operators that commute with G. A commuting set of these operators can be chosen to be Hermitian and their eigenvalues are used to label the multiplicity. Moreover, the intertwining operators that map an irreducible G-module into V also intertwine the generalized Casimir operators.

Invariant theory comes into play when the representation contragredient to the G-module is tensored with the H module. The invariant subspace of this augmented space—that is, the subspace of elements invariant under the G action—has dimension equal to the multiplicity.

If G and H have duals G' and H', respectively, then the generalized Casimir operators can be written in terms of the dual actions; that is, the set of generalized Casimir operators is equivalently defined as the elements of the universal enveloping algebra of G' that commute with H'. Stated in this way the generalized Casimir operators act naturally on the invariant subspace, and in fact leave it invariant. Therefore, the eigenvectors of a complete commuting set will form an orthonormal basis in the invariant subspace. Different choices of complete commuting sets will result in different sets of orthonormal bases and their overlap we have called invariant coefficients.

The main goal of this paper has been to apply this set-up to the $U(N)$ groups and show how it is linked to the decomposition of n-fold tensor products. Then H is the outer product group $U(N) \times \cdots \times U(N)$ and G is the restriction to the diagonal subgroup $U(N)$. Irreducible representations of G are realized as polynomials on Fock space as is the n-fold tensor product space. The n-fold tensor product of $U(N)$ irreps is irreducible under H, but becomes reducible under the restriction to G. The eigenvalues of generalized Casimir operators, operators from the universal enveloping algebra of H that commute with G, then break the multiplicity.

To gain greater insight into the multiplicity structure and to make the multiplicity-breaking procedure more computationally effective, the n-fold tensor product space is augmented by the representation contragredient to the $U(N)$ representation of interest. Though the contragredient is defined via linear functionals, we show (in the appendix) that for $U(N)$ there is an equivalent definition given in terms of the so-called 'check' representation, defined in equation (2.15), which is again a polynomial representation in the Fock space. Thus, the n-fold tensor product space tensored with the contragredient representation is a subspace of a polynomial space denoted by $\mathcal{H}^{(m)\otimes(M)^{\surd}}$ (see equation (2.14)); the invariant subspace $\mathcal{J}^{(m),(M)^{\surd}}$ is then the subspace of $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$ of vectors that are invariant under the $U(N)$ action.

Both G and H have duals on the Fock space. If (m) is the set of integers specifying irreps in the n-fold tensor product (with all the zeros deleted), then G' is the group $U(p)$ (or $GL_p(\mathbb{C})$), where p is the number of entries in (m) and H' is $U(p_1) \times \cdots \times U(p_r)$, $\sum_{k=1}^{r} p_k = p$. The generalized Casimir operators, originally defined with respect to H and G, are shown to be equivalently defined as elements in the universal enveloping algebra of G' that commute with H' (see theorem 3.1). Proposition 3.3 then shows that the so-defined generalized Casimir operators leave the space $\mathcal{J}^{(m),(M)^{\surd}}$ invariant.

By choosing a complete commuting set of Hermitian generalized Casimir operators, an orthonormal basis of eigenvectors in $\mathcal{J}^{(m),(M)^{\surd}}$ can be constructed, and different choices of complete commuting sets will give different orthonormal bases, the overlaps of which are the invariant coefficients. If the generalized Casimir operators are chosen as those Casimir operators arising from a choice of coupling the n-fold tensor products in a definite sequence, this set is then complete and commuting. If another coupling sequence is chosen, another set of orthonormal bases will result. The overlaps of such bases in $\mathcal{J}^{(m),(M)\surd}$, resulting from different coupling sets, are usually called Racah coefficients; from the perspective of this paper Racah coefficients are special cases of invariant coefficients, in which the generalized Casimir operators are chosen from different coupling schemes in the tensor product.

To illustrate how these results can be practically implemented, we have written Mathematica programs, whose content is given in [5]. Using these programs, in section 4 we have shown how to decompose the eight-dimensional representation (2, 1, 0) of $U(3)$, tensored with itself three times. The (5, 3, 1) representation has a multiplicity of 6, and we exhibit the six eigenvectors that arise from different sets of commuting generalized Casimir operators. Since the eight-dimensional representation of $U(3)$ is tensored with itself three times, there is also a permutation symmetry that can be used to label the multiplicity and generate the six eigenvectors. The overlap between these different sets of eigenvectors is given in table 1, and includes Racah coefficients for different stepwise couplings.

As argued in the beginning of this conclusion the setup we have applied to the $U(N)$ groups can be applied to many different groups and subgroups. For example, the restriction of $U(N)$ irreps to $SO(N)$ irreps is well known to have multiplicity, which has been dealt with when $N = 3$ by introducing generalized Casimir operators. The main difficulty in actually implementing the program outlined here is in finding complete commuting sets of generalized Casimir operators. We know of no general procedure by which such commuting sets can be exhibited.

Acknowledgments

WHK and TTT wish to thank the Obermann Center of the University of Iowa for support while completing this paper.

Appendix

Let $R_{ij}^{(M)}$, $1 \leqslant i, j \leqslant N$, denote the infinitesimal operators of $R^{(M)}$ corresponding to the standard basis $\{e_{ij}\}$ of $\mathbb{C}^{N\times n}$, i.e.,

$$R_{ij}^{(M)} = \frac{\mathrm{d}}{\mathrm{dt}} R^{(M)}(I + te_{ij})|_{t=0} \qquad i \neq j$$

and

$$R_{ii}^{(M)} = \frac{\mathrm{d}}{\mathrm{dt}} R^{(M)}(I + (t-1)e_{ii})|_{t=0}.$$

Then an easy computation shows that

$$R_{ij}^{(M)} = \sum_{\gamma=1}^{n} Z_{\gamma i} \frac{\partial}{\partial Z_{\gamma j}}.$$

In the appendix of [2] it was shown that

$$R_{ij}^{(M)\dagger} = R_{ji}^{(M)}$$

where $R_{ij}^{(M)\dagger}$ denotes the adjoint of the operator $R_{ij}^{(M)}$. This also means that

$$R_{ij}^{(M)} = \frac{\mathrm{d}}{\mathrm{d}t} R^{(M)}(\exp(te_{ij}))\,|_{t=0}.$$

Let $\{h_\xi\}$ be an ONB of $V^{(M)}$, then for $i \neq j$

$$\exp(te_{ij}) = I + te_{ij} = \begin{array}{cc} & \begin{array}{ccccc} & & & j & \end{array} \\ i & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ - - - & & 1 & - - t & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array}.$$

Now $R^{(M)}(\exp(te_{ij})) = \exp\left(tR_{ij}^{(M)}\right)$ (actually since R_{ij} is a nilpotent operator on $V^{(M)}$, $\exp\left(tR_{ij}^{(M)}\right)$ is a polynomial in $R_{ij}^{(M)}$). It follows that

$$\left\langle R^{(M)}(\exp(te_{ij}))h_\xi \mid h_\eta \right\rangle = \left\langle h_\xi \mid R^{(M)\dagger}(\exp(te_{ij}))h_\eta \right\rangle = \left\langle h_\xi \mid \exp\left(tR_{ij}^{(M)}\right)^\dagger h_\eta \right\rangle$$
$$= \left\langle h_\xi \mid \exp\left(tR_{ji}^{(M)}\right) h_\eta \right\rangle = \left\langle h_\xi \mid R^{(M)}(\exp(te_{ji}))h_\eta \right\rangle.$$

Similarly, it can be shown that

$$\left\langle R^{(M)}(\exp(te_{ii}))h_\xi \mid h_\eta \right\rangle = \left\langle h_\xi \mid R^{(M)}(\exp(te_{ii}))h_\eta \right\rangle$$

where

$$\exp(te_{ii}) = \begin{array}{cc} & \begin{array}{cccc} & & i & \end{array} \\ i - & \begin{pmatrix} 1 & & | & & \\ & \ddots & | & & \\ - - - & & e^t & & \\ & & | & \ddots & \\ & & | & & 1 \end{pmatrix} \end{array}.$$

It follows from a well-known theorem (see, e.g., [9, corollary to theorem 2, section 14])

$$\left\langle R^{(M)}(g)h_\xi \mid h_\eta \right\rangle = \left\langle h_\xi \mid R^{(M)}(g^T)h_\eta \right\rangle \tag{A.2}$$

for all ξ and η. And in general

$$\left\langle R^{(M)}(g)f \mid f' \right\rangle = \left\langle f, R^{(M)}(g^T)f' \right\rangle \qquad \forall f, f' \in V^{(M)}.$$

A concrete realization of the contragredient representation to $\left(R^{(M)}, V^{(M)}\right)$ as a submodule of $\mathcal{F}(\mathbb{C}^{n\times N})$ can be given as follows:

Let $\{f_\nu\}$ denote the normalized Gelfand–Cetlin basis for $V^{(M)}$ (see [10] and [11, chapter X]). If $f \in V^{(M)}$ then $f = \sum_\nu c_\nu f_\nu$, and $f^* = \sum_\nu \bar{c}_\nu f_\nu^*$. Thus $f^* \in V^{(M)}$, and it follows immediately that the map $f \to f^*$, $f \in V^{(M)}$, is an *involutory conjugate-linear (anti) automorphism* of $V^{(M)}$, and the Hilbert space $V^{(M)}$ is identified with its dual under this map. Define the representation $R^{(M)^*}$ of $GL_N(\mathbb{C})$ on $V^{(M)}$ as follows: for $f \in V^{(M)}$, $g \in GL_N(\mathbb{C})$, $g^\surd = (g^{-1})^T$, set

$$R^{(M)^*}(g)f^*\colon = \left(R^{(M)}(g^{\checkmark})f\right)^* = \left[\left(R^{(M)}(g^{\checkmark})\right)\left(\sum_\nu C_\nu f_\nu\right)\right]^*$$
$$= \left[\sum_\nu C_\nu R^{(M)}(g^{\checkmark})f_\nu\right]^* = \left[\sum_\nu C_\nu \sum_\mu \left\langle R^{(M)}(g^{\checkmark})f_\nu \mid f_\mu\right\rangle f_\mu\right]^*$$

and (by equation (A.2))

$$= \left[\sum_{\mu,\nu} C_\nu \left\langle \overline{R^{(M)}(g^{-1})f_\mu \mid f_\nu}\right\rangle f_\mu\right]^* = \sum_{\mu,\nu} \bar{C}_\nu \left\langle R^{(M)}(g^{-1})f_\mu \mid f_\nu\right\rangle f_\mu^*. \tag{A.3}$$

Setting $D_{\nu\mu}(g) = \left\langle R^{(M)}(g)f_\mu \mid f_\nu\right\rangle$ we get

$$R^{(M)^*}(g)f^* = \sum_{\mu,\nu} \bar{C}_\nu D_{\mu,\nu}(g^{-1})f_\mu^*.$$

Thus $R^{(M)^*}(g)f^*$ belongs to $V^{(M)}$ if $f \in V^{(M)}$.

If $c \in \mathbb{C}$ then

$$R^{(M)^*}(g)(cf^*) = R^{(M)^*}(g)(\bar{c}f)^* = \left[R^{(M)}(g^{\checkmark})(\bar{c}f)\right]^* = \left[\bar{c}\left(R^{(M)}(g^{\checkmark})f\right)\right]^*$$
$$= c\left[R^{(M)}(g^{\checkmark})f\right]^* = cR^{(M)^*}(g)f^*.$$

If $f_1, f_2 \in V^{(M)}$ then

$$R^{(M)^*}(g)(f_1^* + f_2^*) = R^{(M)^*}(g)(f_1 + f_2)^* = \left[R^{(M)}(g^{\checkmark})(f_1 + f_2)\right]^*$$
$$= \left[R^{(M)}(g^{\checkmark})f_1 + R^{(M)}(g^{\checkmark})f_2\right]^* = \left(R^{(M)}(g^{\checkmark})f_1\right)^* + \left(R^{(M)}(g^{\checkmark})f_2\right)^*$$
$$= R^{(M)^*}(g)f_1^* + R^{(M)^*}(g)f_2^*.$$

Therefore, $R^{(M)^*}(g)$ is a linear operator on $V^{(M)}$. For $g_1, g_2 \in GL_N(\mathbb{C})$

$$R^{(M)^*}(g_1)\left(R^{(M)^*}(g_2)f^*\right) = R^{(M)^*}(g_1)\left(R^{(M)}\left(g_2^{\checkmark}\right)f\right)^* = \left[R^{(M)}\left(g_1^{\checkmark}\right)\left(R^{(M)}\left(g_2^{\checkmark}\right)f\right)\right]^*$$
$$= \left[R^{(M)}\left(g_1^{\checkmark}g_2^{\checkmark}\right)f\right]^* = \left[R^{(M)}((g_1g_2)^{\checkmark})f\right]^* = R^{(M)^*}(g_1g_2)f^*.$$

Therefore, $R^{(M)^*}$ is a representation of $GL_N(\mathbb{C})$ on $V^{(M)}$.

Let $\{h_\xi\}$ be an ONB of $V^{(M)}$ then since $\langle h_\eta \mid h_\xi\rangle = \delta_{\xi\eta}$ it follows that $\{h_\xi^*\}$ is also an ONB of $V^{(M)}$. For every ξ, $h_\xi = \sum_\nu C_\nu^\xi f_\nu$ for some $C_\nu^\xi \in \mathbb{C}$, thus $h_\xi^* = \sum_\nu \bar{C}_\nu^\xi f_\nu^*$. Hence,

$$R^{(M)^*}(g)h_\xi^* = \left[R^{(M)}(g^{\checkmark})h_\xi\right]^*$$

(by equation (A.3))

$$= \sum_{\mu,\nu} \bar{C}_\nu^\xi D_{\nu,\mu}^{(M)}(g^{-1})f_\mu^*.$$

If $h_\eta = \sum_\lambda C_\lambda^\eta f_\lambda$ then $h_\eta^* = \sum_\lambda \bar{C}_\lambda^\eta f_\lambda^*$, and

$$\left\langle R^{(M)^*}(g)h_\xi^* \mid h_\eta^*\right\rangle = \left\langle \sum_{\mu,\nu} \bar{C}_\nu^\xi D_{\nu,\mu}^{(M)}(g^{-1})f_\mu^* \,\middle|\, \sum_\lambda \bar{C}_\lambda^\eta f_\lambda^*\right\rangle$$
$$= \sum_{\mu,\nu,\lambda} \bar{C}_\lambda^\eta C_\nu^\xi\, \overline{D_{\nu,\mu}^{(M)}(g^{-1})}\,\delta_{\mu\lambda} = \sum_{\mu,\nu} \bar{C}_\mu^\eta C_\nu^\xi\, \overline{D_{\nu,\mu}^{(M)}(g^{-1})}$$

whereas,

$$\left\langle R^{(M)}(g^{-1})h_\eta \mid h_\xi \right\rangle = \left\langle R^{(M)}(g^{-1}) \left(\sum_\mu C^\eta_\mu f_\mu\right) \middle| \sum_\lambda C^\xi_\lambda f_\lambda \right\rangle$$
$$= \left\langle \sum_\mu C^\eta_\mu \left(\sum_\nu D^{(M)}_{\nu\mu}(g^{-1}) f_\nu\right) \middle| \sum_\lambda C^\xi_\lambda f_\lambda \right\rangle = \sum_{\mu,\nu,\lambda} \bar{C}^\eta_\mu C^\xi_\lambda \overline{D^{(M)}_{\nu\mu}(g^{-1})}\delta_{\nu\lambda}$$
$$= \sum_{\mu,\nu} \bar{C}^\eta_\mu C^\xi_\nu \overline{D^{(M)}_{\nu\mu}(g^{-1})}\delta_{\nu\lambda} = \sum_{\mu,\nu} \bar{C}^\eta_\mu C^\xi_\nu \overline{D^{(M)}_{\nu\mu}(g^{-1})}.$$

Thus

$$\left\langle R^{(M)^*}(g)h^*_\xi \mid h^*_\eta \right\rangle = \left\langle R^{(M)}(g^{-1})h_\eta \mid h_\xi \right\rangle.$$

This means that if $\mathcal{B}$ (resp. $\mathcal{B}^*$) denotes the basis $\{h_\xi\}$ (resp. $\{h^*_\xi\}$), then

$$\left[R^{(M)^*}(g)\right]_{\mathcal{B}^*} = \left[R^{(M)}(g^{-1})\right]^T_{\mathcal{B}} = \left[\left(R^{(M)}(g)\right)^{-1}\right]^T_{\mathcal{B}}.$$

This shows that the representation $R^{(M)^*}$ on $V^{(M)}$ is indeed isomorphic to the contragredient representation of $R^{(M)}$ on $V^{(M)}$.

If $(M) = (M_1, \ldots, M_n, 0, \ldots, 0)$, let $f^{(M)}_{\max}$ denote the highest weight vector of the normalized Gelfand–Cetlin basis of $V^{(M)}$, then

$$f^{(M)}_{\max}(Z) = C_{\max}\Delta_1^{M_1-M_2}(Z)\ldots\Delta_n^{M_N}(Z)$$

where $C_{\max}$ is a positive scalar. Then clearly $f^{*(M)}_{\max} = f^{(M)}_{\max}$. For

$$d = \begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{NN} \end{pmatrix}$$

then equation (A.2) implies that

$$R^{(M)^*}(d) f^{*(M)}_{\max} = \sum_\mu \left\langle R^{(M)}(d^{-1}) f_\mu \;\middle|\; f^{(M)}_{\max} \right\rangle f^*_\mu = \sum_\mu \left\langle f_\mu \;\middle|\; R^{(M)}((d^{-1})^T) f^{(M)}_{\max} \right\rangle f^*_\mu$$
$$= \sum_\mu \left\langle f_\mu \;\middle|\; d_{11}^{-M_1}\ldots d_{nn}^{-M_n} f^{(M)}_{\max} \right\rangle f^*_\mu = d_{11}^{-M_1}\ldots d_{nn}^{-M_n} f^{*(M)}_{\max}.$$

For b' belonging to the unipotent upper triangular subgroup of $GL_N(\mathbb{C})$ then

$$R^{(M)^*}(b') f^{*(M)}_{\max} = \sum_\mu \left\langle f_\mu \;\middle|\; R^{(M)}((b')^{\surd}) f^{(M)}_{\max} \right\rangle f^*_\mu = f^{*(M)}_{\max}$$

since $(b')^{\surd}$ belongs to the unipotent lower triangular subgroup of $GL_N(\mathbb{C})$. It follows that $f^{*(M)}_{\max} \equiv f^{(M)}_{\max}$ is the lowest weight vector of $(R^{(M)^*}, V^{(M)})$ with lowest weight $(-M_1, \ldots, -M_n, 0, \ldots, 0)$ (recall that the signature of the representation $(M)^{\surd}$ is $(0, \ldots, 0, -M_n, \ldots, -M_1)$). But according to [12], a concrete realization of an irreducible $GL_N(\mathbb{C})$-module with signature $\underbrace{(0, \ldots, 0, -M_n, \ldots, -M_1)}_N$ can be defined on $V^{(M)}$ by setting

$$\left[R^{(M)^{\surd}}(g) f\right](Z) = f(Zg^{\surd}) \qquad \forall f \in V^{(\mu)} \quad g \in GL_N(\mathbb{C}).$$

If $\Phi : V^{(M)}$ is defined by $\Phi(f) = f^*$, then equation (A.3) shows that

$$R^{(M)^*}(g)\Phi(f) = \Phi\left(R^{(M)^{\surd}} f\right) \qquad \forall f \in V^{(M)}.$$

Thus Φ is an anti-automorphism intertwining $R^{(M)\surd}$ and $R^{(M)^*}$. It follows that under this identification the $GL_N(\mathbb{C})$-module contragredient to $\left(R^{(M)}, V^{(M)}\right)$ is indeed $\left(R^{(M)\surd}, V^{(M)}\right)$. It follows that $\mathcal{H}^{(m)} \otimes V^{(M)\surd}$ can be defined as

$$\mathcal{H}^{(m)} \otimes V^{(M)\surd} = \left\{ f \in \mathcal{P}(\mathbb{C}^{n\times N}) : f\begin{pmatrix} \beta Z \\ bW \end{pmatrix} = \pi^{(m)}(\beta)\pi^{(M)}(b) f\begin{pmatrix} Z \\ W \end{pmatrix} \right\}$$

where β is defined by equation (2.12) and $b \in B_q$.

References

[1] Klink W H and Ton-That T 1996 Multiplicity invariants, and tensor products of compact groups *J. Math. Phys.* **37** 6468

[2] Klink W H and Ton-That T 1988 On resolving the multiplicity of arbitrary tensor products of $U(N)$ groups *J. Phys. A: Math. Gen.* **21** 3877

[3] Weyl H 1946 *The Classical Groups, Their Invariants and Representations* 2nd edn (Princeton, NJ: Princeton University Press)

[4] Aulwes R 1999 Computational methods in dual representations and invariant theory *Ph D Dissertation* University of Iowa, USA

[5] Aulwes R, Klink W H and Ton-That T 1999 *Preprint*

[6] Klink W H and Ton-That T 1989 n-fold tensor products of $GL(N, \mathbb{C})$ and decomposition of Fock spaces *J. Func. Anal.* **84** 1

[7] Klink W H and Ton-That T 1992 Duality in representation theory *Ulam Quart.* **1** 41

[8] Klink W H and Ton-That T 1990 Representations of $S_n \times U(N)$ in repeated tensor products of the unitary groups *J. Phys. A: Math. Gen.* **23** 2751

[9] Godement R 1982 *Introduction à la theórie des groupes de Lie* (Paris VII, France: Publication Math. de l' Université)

[10] Gelfand I M and Graev M I 1965 Finite-dimensional irreducible representations of the full linear groups, and related special functions *Izv. Akad. Nauk SSSR, Ser. Math.* **29** 1329, 1356 (1967 *AMS Transl. Ser. 2* **64** 116–46)

[11] Zelobenko D P 1973 *Compact Lie Groups and Their Representations* vol 40 (Engl. transl. 1973 *Transl. Math. Monographs* (Providence, RI: AMS))

[12] Ton-That T 1995 Dual representations and invariant theory *Contemp. Math.* **191** 205